Carnap's Conventionalism in Geometry

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Carnap suggests in the *Aufbau* that Kant's division of judgments into synthetic a priori and other variants of synthetic/analytic and a priori/a posteriori judgments can be completely replaced by the conventional and the empirical (see [2, 289], §179). Carnap often justifies his conventionalism with respect to language and logic by analogy to conventionalism in geometry. Expressing a proposition in a natural language is analogous to expressing a topological fact in a conventional metric. Translating natural language sentences from German to French, for example, compares to translating a statement belonging to one metrical spatial form into another (see [1, 99]).

Thomas Mormann's contention is that, whatever may be true about conventionalism in general, the mathematical discipline of differential topology does not support conventionalism in geometry. There are higher-level objections to Carnap's conventionalism in geometry, for example by Quine, who considers it incompatible with holism, and by Ryckman or Friedman, who consider it incompatible with Einstein's theory of relativity. Mormann argues that the mathematical problems of Carnap's account render those higher-level objections unnecessary. We raise objections to Mormann's argument and claim that, whatever else may be said about Carnap's conventionalism in geometry, it does not run afoul of mathematical topology.

Let us agree on the convention that the Earth's surface has zero curvature everywhere. Mormann's topological interpretation of this claim is (where S^2 is the surface of a two-dimensional sphere)

$$S^2 \text{ can be endowed with a metric } l_1 \text{ with curvature } K = 0 \qquad \text{(C1)}$$

This topology, according to Carnap, does not contradict any geodetic measurements or physical observations. He is not quite happy with this example, however, because the metric l_1 must give preference to a particular point in S^2. This preference does not sit well with our requirement for simplicity. Instead of postulating curvature $K = 0$ everywhere, we have the choice of postulating $K = k$ everywhere, where $k > 0$ is the curvature corresponding to the curvature of S^2 given l_0, the Euclidean distance measure we are used to. Now we no longer need a privileged point to define a distance measure l_2 for this topology (also extending it from S^2 to $\mathbb{R}^3$), which has a positive curvature $k > 0$ everywhere:

$$l_2(A, B) = l_0(A, B)(1 - \sin h)$$

We need a postulate on how to measure h, which Carnap provides with the following rule:

$$\int_0^h \frac{1}{1 - \sin x} dx = a$$

where a is the length of a measuring rod measuring h transferred to S^2. Again, Carnap claims that accepting this topology and metric will not put us at odds with any empirical observations or measurements. Mormann translates his claim in terms of differential topology into

$$\begin{array}{l}\mathbb{R}^3 \text{ can be endowed with a metric } l_2 \\ \text{with constant positive curvature } K = k\end{array} \tag{C2}$$

Mormann now provides a proof that, under suitable conditions, both (C1) and (C2) are false.

For polyhedra, the Euler-Poincaré characteristic $\chi(T)$ is known as the number of vertices minus the number of edges plus the number of areas. The theorem of Gauss-Bonnet states that for a compact two-dimensional Riemannian manifold M without a boundary (such as S^2), the total Gaussian curvature is (A being the area element of M)

$$\iint_M K dA = 2\pi\chi(M)$$

The Euler-Poincaré characteristic for an orientable compact surface homeomorphic to a sphere with some handles attached is $2-2g$, g being the number of handles. Consequently, $\chi(S^2) = 2$, and (C1) is false.

Now let M be a complete connected Riemannian manifold with curvature $K \geq a > 0$ (call this last condition (*)). Bonnet's theorem states that then

M must be compact. Because $\mathbb{R}^3$ fulfills all these conditions except (*) and is not compact, (C2) is false. (Both of these proofs see [4, 820f].)

What Mormann initially hides in footnotes (footnote 9 and footnote 12) and eventually discusses in a section toward the end of his article is that his idealized mathematical conditions do not necessarily match the pragmatic constraints Carnap assumes to be true for the physicists doing the work of finding empirical disconfirmation of physical theories with respect to applicable conventions.

Mormann clearly disagrees with Carnap on the admissibility of limitation in empiricist inquiry. This disagreement, somewhat obscure in Mormann's article, explains their mathematical disagreement. (C1) and (C2) are not false, Carnap just never makes clear that he admits limitations and the Riemannian manifolds may not be complete (a space X is complete if every Cauchy sequence in it converges). Mormann complains that completeness is "indispensable from an empiricist point of view" [4, 817], that incompleteness "lacks empirical significance" [4, 820], that "it would be a desperate move to attempt to rescue Carnap's thesis by allowing him to fall back on incomplete metrics" [4, 821], and, most relevantly, that

> for an empiricist it is meaningless to be engaged in investigating the global structure of the world under the presupposition that large areas of that world are principally inaccessible to empirical investigation. (Mormann [4, 823])

In reply to Mormann, first off we need to note that completeness is not the issue for (C1). Let a plane F go through a point on the radius between the centre of the Earth and the North Pole (say 6000km away from the centre of the Earth) and be parallel to the equatorial plane. Then define T^2, think of it as a punch bowl or a spherical decapitated eggshell, as the intersection of $\mathbb{R}^3$ south of F (including F) and S^2. T^2 fulfills the conditions of the Gauss-Bonnet theorem, and there is no longer a problem with Carnap's claim that T^2 can be endowed with a metric whose curvature is 0, as Gauss-Bonnet's theorem for a space with a boundary runs like this (see [3, 260]):

$$\int_{\partial M} k_g ds + \iint_M K dA = 2\pi - \sum_{j=1}^{m} \alpha(p_j)$$

where ∂M is the boundary of M, k_g is the geodesic curvature of ∂M, and the $\alpha(p_j)$ are the exterior angles of the corners $p_1, \ldots, p_m$ of ∂M.

Foreboding as this formula may look it actually makes good sense. Our boundary (the intersection of F and S^2) has no corners, so we can ignore the sum of exterior angles. The concavity of the boundary, however, makes up for the convexity of the sphere so that it is possible to endow T^2 with a metric with constant curvature $K = 0$. You may ask why we did not keep T^2 open and exclude the boundary, which would also provide us with the possibility of a metric with constant curvature $K = 0$. Such a space would be homeomorphic to $\mathbb{R}^2$, very close to what Carnap had in mind, but it lacks the completeness we were hoping for. In any case, T^2 as defined is complete and fulfills Carnap's criteria.

Thus, when Mormann says that with incomplete metrics, while "(C2) could be saved, (C1) remains false" [4, 821], it remains false because we do not even need to go as far as retreating to incomplete metrics. We can keep (C1) by introducing a boundary, or, as Carnap would say, a limitation. It is not so much mathematical inconsistency that is at the heart of this problem, but rather a lack of clarity to what extent the limitations of scientific observation enter into which questions it is in principle possible to answer. Our impression, unfortunately not based on a clarification by Carnap himself, is that he includes practical limitations in his account of the limits of science. To pose a question, Carnap says in §180 of the *Aufbau*, "is to give a statement together with the task of deciding whether this statement or its negation is true" [2, 290]. If the task is 'in principle' impossible to carry out, which it very well may be (unless 'in principle' means just the opposite of 'in practice'), then it remains open whether the question is properly posed.

References

[1] Carnap, R., 1978. Der Raum: Ein Beitrag zur Wissenschaftslehre. *Kant-Studien* Ergänzungsheft 56. Berlin: Reuther und Reichard. Translated by Michael Friedman and Peter Heath as *Space. A Contribution to the Theory of Science*. Unpublished.

[2] Carnap, R., 2003. *The Logical Structure of the World and Pseudoproblems in Philosophy (Open Court Classics)*. Open Court Publishing Company.

[3] Chavel, I., 2006. *Riemannian Geometry: A Modern Introduction*. Cambridge University.

[4] Mormann, T., 2004. Carnap's Metrical Conventionalism Versus Differential Topology. *Philosophy of Science*, 72(5):814–825.